From students' processes and classroom practises to competencies in mathematics.

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In his plenary lecture at the second ICME René Thom wrote: "The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of *meaning*" (Thom, 1972: p. 202). More specifically, the *real problem* consists in focusing the genesis of mathematical objects in the classroom. To do that, many tools are necessary, some of which appear far from the formal mathematical frame: from epistemology to neurology, through history, psychology, ergonomy.
The main goal of my lecture consists in discussing some of them. For reasons of time and clearness, the discussion will be exemplified on some concrete cases, mainly concerning the concept of function (in a wide sense of the word, from the geometrical transformations to the functions of Calculus). Each example shows a different lens for designing/analysing the genesis of the mathematical concepts in the classroom. Moreover they possibly show how to fill up the gap between the *worldly truth* in which our students make their concrete experiences and the *logical truth*, which represents the rigorous official side of mathematics. In the end they will result in a more complex landscape, where the two aspects will not appear so dramatically contrasted and where mathematical objects do live. In fact our mind's activity depends upon an integrated and dynamic set which acts as a whole, whose components are: the brain, the body, the cultural and physical world. When students learn mathematics all those components (and possibly others, e.g. emotional ones) are active and must be taken into consideration by the teacher.
As far as functions are concerned, the existing literature has analysed many difficulties in students, e.g. the incapability of understanding properly the sense of the graph in its global features, or the variational and co-variational aspects of variables; the mismatching between the graph in space-time diagrams and the route in the space, or between the height and the slope of the graph, etc. Each difficulty is related to a piece of the function's meaning, which generally is rooted in many cultural experiences within the history of mathematics: to make an example, some of them involved instruments to draw curves, with the recipes to manage them. In the end mathematicians reified them in signs and symbols and contracted them in the 'mathematical' definition of function as a set of ordered pair in a cartesian product of two sets (Bourbaki, 1939, p. 76). There is a big gap between the early notion of function (e.g. in Newton or in Euler) and the abstract definition based on set theory. In fact, the former is based on relationships between concrete, dynamic and continuous variables, where the motion and the idea of change are triggering and supporting the main concepts. On the contrary, the latter introduces the pure mathematical skeleton of the notion, distilled in the course of centuries and the student risks to get lost in it. Nowadays, the existence of suitable technological instruments puts forward the old cognitive and cultural roots of functions as objects which represent how things change: e.g. in dynamic geometry software or in probe instruments that can collect data of moving objects and people in real time, and represent them suitably on the screen of a computer.
This phenomenon is not only linked to the considered example, namely functions, but has a general feature.
Today's new information and communication devices are the origin of what N. Balacheff (1997, p. 113) calls "a new semiotics of mathematics" (e.g.: the geometry of ruler and compass, of Logo, of Cabri with respect to the 'official' Euclidean or transformational

geometry; the approach to concept of function, sketched above; and so on). Namely, new technology can be the agent of a redefinition of knowledge, in particular of school knowledge. Specifically for mathematics this issue implies the restructuring of the role of notations and symbols and consequently the meaning of concepts (e.g. through the *instrumental genesis* of concepts, in the sense of Rabardel, 1995).

In fact, the new technology that enters the classroom must be analysed in a wide cultural sense. An interesting study in this direction has been developed by J. Kaput, R. Noss & C. Hoyles (2002), who introduced the notion of *representational infrastructure*:

> *The appearance of new computational forms and literacies is pervading the social and economic lives of individuals and nations alike.…*
> *The real changes are not technical, they are cultural. Understanding them… is a question of the social relations among people, not among things...*
> *The notational systems we use to present and re-present our thoughts to ourselves and to others, to create and communicate records across space and time, and to support reasoning and computation constitute a central part of any civilization's infrastructure. As with infrastructure in general, it functions best when it is taken for granted, invisible, when it simply "works".*

This implies to analyse Information and Communication Technology in the classroom from two points of view:

a) as *Cultural Semiotic Systems*, namely as cultural systems which make available varied sources for meaning-making through specific social signifying practises (a concept which is due to L. Radford, 2003); such practises are not (only) to be considered within the strictly school environment but within the larger environment of the whole society, embedded in the stream of its history;

b) as *Intrinsic Cognitive Energizers*, that is as media which have *intrinsically* a logic which enters in cognitive resonance with the subjects (an idea which comes in the long run from the notion of *intuitive model* of E. Fischbein, 1987).

This makes it possible for the teacher to design and manage learning contexts that involve components of a different nature, for example bodily and culturally based.

Some examples from our researches will illustrate the complex interplay among cultural, biological and cognitive aspects in the *objectification processes* (Radford, 2002) that accompany the production of mathematical ideas within different specific contexts, where new technologies are present (e.g. when functions are used for modelling purposes). The examples are analysed according to different tools: epistemology, semiotics, psychology, neurology. They allow to point out different ingredients in the objectification processes of mathematical entities, e.g. functions. Such ingredients may condense in what I call the *cognitive space of action, production and communication* (Arzarello, in print). The APC-Space is built up, developed and shared in the classroom. Its main components are:

- the body;
- the physical world;
- the cultural environment.

APC spaces are the didactical counterpart of representational infrastructures. Namely APC components can describe suitably learning processes within a system, which has been designed according the two dimensions (a and b above) of a representational infrastructure. The APC-space is built up in the classroom as a dynamic single system, where the different components are integrated each other into a whole unity. The integration is a product of the interactions among pupils, the mediation of the teacher and the interactions with artefacts. The three letters A, P, C illustrate its dynamic features, namely the fact that three main components characterise learning mathematics: students' actions and interactions (with their mates, with the teacher, with themselves, with tools), their productions (e.g. answering a question, posing other questions, and so on) and communication aspects (e.g. when the discovered solution is communicated to a mate or to the teacher, using suitable representations).

In the presentation I'll use the notion of APC space and of representational infrastructure can be used to illustrate a teaching experiment that our research team in Turin (L. Bazzini, O. Robutti, D. Paola, F. Ferrara, C. Sabena) is developing since a couple of years, where functions and pre-calculus are approached in secondary school starting from grade 9 with a systematic use of different technological environments.

I'll sketch both the a-priori analysis, which entails both the *embodied* and the *cultural* nature of function's concept, and some findings that we have drawn. I'll show that the APC-space model allows properly studying the so called *perceptuo-motor* features in the processes of knowing (Antinucci, 2001; Nemirovsky & Borba, 2003), which reveal crucial for learning in technological environments. Namely, it allows to illustrate how action and perception determine the processes of learning and to describe them so that doing, touching, moving and seeing appear as their important ingredients. I'll show how a learning approach based on perceptuo-motor activities requires suitable modalities of teaching, in which the students are actively involved in the construction of mathematical concepts.

The findings in our teaching experiment, with all their limits, confirm that this approach is a useful research tool to understand the ways in which technological artefacts can mediate/support the construction of the student's mathematical knowledge. In fact, the analysis of the cultural and cognitive ingredients of the technical tools used in the classroom allows to consider the value-added component provided by the technology not limited to its purely technical features. As well, the APC-space components allow to consider the emergence of the expected knowledge not as a result of a game definitely confined to the relationships between the subjects and the 'milieu' (Brousseau, 1997) but in an environment where the 'game' consists in a semiotic mediation that involves the students, the ICT and the teacher, who rules and supports the evolution of the personal senses, which students attach to their actions with ICT, towards the scientific shared sense. The teacher's task consists in promoting the integration of the cultural and biological roots of the mathematical ideas within suitable representational infrastructures. This approach allows to nurture their cognitive resonance in students and produce what I call *learning in a natural setting* (the idea is from Tall, 2001).

The space of action, production and communication goes beyond the Vygotskian rigid distinction between technical tools and psychological tools (Vygotsky, 1978, p. 53). In fact recent neurological research (e.g. Gallese, 2003) gives a picture of representational and conceptual content as the result of the ongoing modelling process of an organism as currently integrated with the object to be represented.

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